

# On Chromatic Number of Kneser Hypergraphs

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## Abstract

In this paper, in view of  $Z_p$ -Tucker lemma, we introduce a lower bound for chromatic number of Kneser hypergraphs which improves Dol'nikov-Kříž bound. Next, we introduce multiple Kneser hypergraphs and we specify the chromatic number of some multiple Kneser hypergraphs. For a vector of positive integers  $\vec{s} = (s_1, s_2, \dots, s_m)$  and a partition  $\pi = (P_1, P_2, \dots, P_m)$  of  $\{1, 2, \dots, n\}$ , the multiple Kneser hypergraph  $KG^r(\pi; \vec{s}; k)$  is a hypergraph with the vertex set

$$V = \{A : A \subseteq P_1 \cup P_2 \cup \dots \cup P_m, |A| = k, \forall 1 \leq i \leq m; |A \cap P_i| \leq s_i\}$$

whose edge set is consist of any  $r$  pairwise disjoint vertices. We determine the chromatic number of multiple Kneser hypergraphs provided that  $r = 2$  or for any  $1 \leq i \leq m$ , we have  $|P_i| \leq 2s_i$ . In particular, one can see that if  $|P_1| = |P_2| = \dots = |P_m| = t$ ,  $m \geq k$ , and  $\vec{s} = (1, 1, \dots, 1)$ , then  $\chi(KG^2(\pi; \vec{s}; k)) = t(m - k + 1)$ . This gives a positive answer to a problem of Naserasr and Tardif [The chromatic covering number of a graph, Journal of Graph Theory, 51 (3): 199–204, (2006)].

A subset  $S \subseteq [n]$  is almost  $s$ -stable if for any two distinct elements  $i, j \in S$ , we have  $|i - j| \geq s$ . The almost  $s$ -stable Kneser hypergraph  $KG^r(n, k)_{\vec{s}\text{-stab}}$  has all  $s$ -stable subsets of  $[n]$  as the vertex set and every  $r$ -tuple of pairwise disjoint vertices forms an edge. Meunier [The chromatic number of almost stable Kneser hypergraphs. J. Combin. Theory Ser. A, 118(6):1820–1828, 2011] showed for any positive integer  $r$ ,  $\chi(KG^r(n, k)_{\vec{s}\text{-stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . We extend this result to a large family of Schrijver hypergraphs. Finally, we present a colorful-type result which confirms the existence of a completely multicolored complete bipartite graph in any coloring of a graph.

**Keywords:** Chromatic Number, Kneser Hypergraph,  $Z_p$ -Tucker Lemma, Tucker-Ky Fan's Lemma.

**Subject classification:** 05C15

## 1 Introduction

In this section, we setup some notations and terminologies. Hereafter, the symbol  $[n]$  stands for the set  $\{1, \dots, n\}$ . A *hypergraph*  $\mathcal{H}$  is an ordered pair  $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ ,

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where  $\mathcal{V}(\mathcal{H})$  (the *vertex set*) is a finite set and  $\mathcal{E}(\mathcal{H})$  (the *edge set*) is a family of distinct non-empty subsets of  $\mathcal{V}(\mathcal{H})$ . Throughout this paper, we suppose that  $\mathcal{V}(\mathcal{H}) = [n]$  for some positive integer  $n$ . Assume that  $N = (N_1, N_2, \dots, N_r)$ , where  $N_i$ 's are pairwise disjoint subsets of  $[n]$ . The *induced hypergraph*  $\mathcal{H}_{|N}$  has  $\cup_{i=1}^r N_i$  and  $\{A \in \mathcal{E}(\mathcal{H}) : \exists i; 1 \leq i \leq r, A \subseteq N_i\}$  as the vertex set and the edge set, respectively. If every edge of hypergraph has size  $r$ , then it is called an *r-uniform* hypergraph. A *t-coloring* of a hypergraph  $\mathcal{H}$  is a mapping  $h : \mathcal{V}(\mathcal{H}) \rightarrow [t] = \{1, 2, \dots, t\}$  such that every edge is not monochromatic. The minimum  $t$  such that there exists a  $t$ -coloring for hypergraph  $\mathcal{H}$  is termed its *chromatic number*, and is denoted by  $\chi(\mathcal{H})$ . If  $\mathcal{H}$  has an edge of size 1, then we define the chromatic number of  $\mathcal{H}$  to be infinite. For any hypergraph  $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$  and positive integer  $r \geq 2$ , the *Kneser hypergraph*  $\text{KG}^r(\mathcal{H})$  has  $\mathcal{E}(\mathcal{H})$  as its vertex set and the edge set consisting of all  $r$ -tuples of pairwise disjoint edges of  $\mathcal{H}$ . It is known that for any graph  $G$ , there exists a hypergraph  $\mathcal{H}$  such that  $G$  is isomorphic to  $\text{KG}^2(\mathcal{H})$ .

A subset  $S \subseteq [n]$  is *s-stable* (reps. *almost s-stable*) if any two distinct elements of  $S$  are at least “at distance  $s$  apart” on the  $n$ -cycle (reps.  $n$ -path), that is,  $s \leq |i - j| \leq n - s$  (reps.  $|i - j| \geq s$ ) for distinct  $i, j \in S$ . Hereafter, for a subset  $A \subseteq [n]$ , the symbols  $\binom{A}{k}$ ,  $\binom{A}{k}_s$ , and  $\binom{A}{k}_s^\sim$  stand for the set of all  $k$ -subsets of  $A$ , the set of all  $s$ -stable  $k$ -subsets of  $A$ , and the set of all almost  $s$ -stable  $k$ -subsets of  $A$ , respectively. One can see that  $\binom{A}{k}_s \subseteq \binom{A}{k}_s^\sim \subseteq \binom{A}{k}$ . Assume that  $\mathcal{H}_1 = (A, \binom{A}{k})$ ,  $\mathcal{H}_2 = (A, \binom{A}{k}_s)$ , and  $\mathcal{H}_3 = (A, \binom{A}{k}_s^\sim)$ . Hereafter, for any positive integer  $r \geq 2$ , the hypergraphs  $\text{KG}^r(\mathcal{H}_1)$ ,  $\text{KG}^r(\mathcal{H}_2)$ , and  $\text{KG}^r(\mathcal{H}_3)$  are denoted by  $\text{KG}^r(A, k)$ ,  $\text{KG}^r(A, k)_{s\text{-stab}}$ , and  $\text{KG}^r(A, k)_{s\text{-stab}}^\sim$ , respectively. Also,  $\text{KG}^r([n], k)$ ,  $\text{KG}^r([n], k)_{s\text{-stab}}$ , and  $\text{KG}^r([n], k)_{s\text{-stab}}^\sim$  are denoted by  $\text{KG}^r(n, k)$ ,  $\text{KG}^r(n, k)_{s\text{-stab}}$ , and  $\text{KG}^r(n, k)_{s\text{-stab}}^\sim$ , respectively. Moreover, hypergraphs  $\text{KG}^r(n, k)$ ,  $\text{KG}^r(n, k)_{s\text{-stab}}$ , and  $\text{KG}^r(n, k)_{s\text{-stab}}^\sim$  are termed the *Kneser hypergraph*, the *s-stable Kneser hypergraph*, and the *almost s-stable Kneser hypergraph*, respectively. In 1955, Kneser [16] conjectured  $\chi(\text{KG}^2(n, k)) = n - 2k + 2$ . Later, Lovász [19], in his fascinating paper, confirmed the conjecture using algebraic topology. Next, Erdős [9] presented an upper bound for the chromatic number of Kneser hypergraphs and conjectured the equality. In [2], this conjecture has been confirmed and it was shown  $\chi(\text{KG}^r(n, k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . In view of result of Erdős [9], one can conclude that for any positive integer  $r \geq 2$ , we have  $\chi(\text{KG}^r(n, k)_{s\text{-stab}}) \leq \chi(\text{KG}^r(n, k)_{s\text{-stab}}^\sim) \leq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . Also, Meunier [24] has shown that when  $s \geq r$ , then  $\chi(\text{KG}^r(n, k)_{s\text{-stab}}) \leq \left\lceil \frac{n-s(k-1)}{r-1} \right\rceil$ . Finding a lower bound for chromatic number of hypergraphs has been studied in the literature, see [7, 17, 18, 28, 29, 34, 35]. As an interesting result, we have Dol’nikov-Kříž bound. For a hypergraph  $\mathcal{H}$ , the *r-colorability defect* of  $\mathcal{H}$ , say  $cd_r(\mathcal{H})$ , is the minimum number of vertices which should be excluded such that the induced hypergraph on the remaining vertices has chromatic number at most  $r$ .

**Theorem A.** (Dol’nikov for  $r = 2$ , Kříž [7, 17]) *For any hypergraph  $\mathcal{H}$  and positive integer  $r \geq 2$ , we have*

$$\chi(\text{KG}^r(\mathcal{H})) \geq \frac{cd_r(\mathcal{H})}{r-1}.$$

Alon et al. [3] constructed ideals in  $\mathbb{N}$  which are not non-atomic but they have the Nikodým property by using stable Kneser hypergraphs. In this regard, they studied the chromatic number of  $r$ -stable Kneser hypergraph  $\text{KG}^r(n, k)_{r\text{-stab}}$  and presented the following conjecture.

**Conjecture A.** [3] *Let  $k, r$  and  $n$  be positive integers where  $n \geq rk$  and  $r \geq 2$ . We have*

$$\chi(\text{KG}^r(n, k)_{r\text{-stab}}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

In [3], it has been shown that the aforementioned conjecture holds when  $r$  is a power of 2. As an approach to Conjecture A, Meunier [24] showed that  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$  and he strengthened the above conjecture as follows.

**Conjecture B.** [24] *Let  $k, r, s$  and  $n$  be positive integers where  $n \geq sk$  and  $s \geq r \geq 2$ . We have*

$$\chi(\text{KG}^r(n, k)_{s\text{-stab}}) = \left\lceil \frac{n - s(k - 1)}{r - 1} \right\rceil.$$

In [15], it has been shown that if  $r = 2$  and  $n$  is sufficiently large, then for  $s \geq 4$  the aforementioned conjecture holds.

This paper is organized as follows. In Section 1, we set up notations and terminologies. Section 2 will be concerned with Tucker's lemma and its generalizations. In Section 3, we introduce the  $i^{\text{th}}$  alternation number of hypergraphs, and in view of  $Z_p$ -Tucker lemma, we present a lower bound for chromatic number of hypergraphs. This lower bound improves the well-know Dol'nikov-Kříž lower bound (Theoreme A). In fact, the alternation number of a hypergraph can be considered as a generalization of colorability defect of hypergraphs. Section 4 is devoted to multiple Kneser hypergraphs. In this section, we determine the chromatic number of some multiple Kneser hypergraphs. In particular, we present a generalization of a result of Alon et al. [2] about chromatic number of Kneser hypergraphs. In Section 5, we extend Meunier's result and it is shown that for any positive integers  $k, n$  and  $r$ , if  $r$  is an even integer or  $n \not\stackrel{r-1}{\equiv} k$ , then for *Schrijver hypergraph*  $\text{KG}^r(n, k)_{2\text{-stab}}$ , we have  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$ . In the last Section, in view of Tucker-Ky Fan's Lemma, we prove a colorful-type result which confirms the existence of a completely multicolored complete bipartite graph in any coloring of a graph in terms of its alternation number.

## 2 Tucker's Lemma and Its Generalizations

In this section, we present Tucker's lemma and some of its generalizations. In fact, Tucker's lemma is a combinatorial version of the Borsuk-Ulam theorem with several interesting applications. For more about Borsuk-Ulam's theorem and Tucker's lemma, see [20].

Throughout this paper, for any positive integer  $r$ , let  $\mathcal{Z}_r = \{\omega_1, \omega_2, \dots, \omega_r\}$  be a set of size  $r$  where  $0 \notin \mathcal{Z}_r$ . Moreover, when  $p$  is a prime integer, we assume that  $\mathcal{Z}_p$  is a *cyclic group* of order  $p$  and *generator*  $\omega$ , i.e.,  $\mathcal{Z}_p = Z_p = \{\omega, \omega^2, \dots, \omega^p\}$ . In particular, when  $r = 2$ , we set  $\mathcal{Z}_2 = Z_2 = \{\omega, \omega^2\} = \{-1, +1\}$ . Consider a *ground set*  $S$  such that  $0 \notin S$  and  $|S| \geq 2$ . Assume that  $X = (x_1, x_2, \dots, x_n)$  is a sequence of  $S \cup \{0\}$ . The subsequence  $x_{j_1}, x_{j_2}, \dots, x_{j_m}$  ( $j_1 < j_2 < \dots < j_m$ ) is said to be an *alternating sequence* if any two consecutive terms in this subsequence are different. We denote by  $\text{alt}(X)$  the size of a longest alternating subsequence of non-zero terms in  $X$ . For instance, if  $S = \mathcal{Z}_4$  and  $X = (\omega_4, 0, \omega_2, \omega_1, 0, \omega_1, \omega_3, \omega_1)$ , then  $\text{alt}(X) = 5$ .

One can consider  $(\mathcal{Z}_r \cup \{0\})^n$  as the set of all *singed subsets* of  $[n]$ , that is, the family of all  $(X^1, X^2, \dots, X^r)$  of disjoint subsets of  $[n]$ . Precisely, for  $X = (x_1, x_2, \dots, x_n) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $j \in [r]$ , we define  $X^j = \{i \in [n] : x_i = \omega_j\}$ . Throughout of this paper, for any  $X \in (\mathcal{Z}_r \cup \{0\})^n$ , we use these representations interchangeably, i.e.,  $X = (x_1, x_2, \dots, x_n)$  or  $X = (X^1, X^2, \dots, X^r)$ . Assume  $X, Y \in (\mathcal{Z}_r \cup \{0\})^n$ . By  $X \preceq Y$ , we mean  $X^i \subseteq Y^i$  for each  $i \in [r]$ . Note that if  $X \preceq Y$ , then any alternating subsequence of  $X$  is also an alternating subsequence of  $Y$  and therefore  $\text{alt}(X) \leq \text{alt}(Y)$ . Also, note that if the first non-zero term in  $X$  is  $\omega_j$ , then any alternating subsequence of  $X$  of maximum length begins with  $\omega_j$  and also, we can conclude that  $X^j$  contains the smallest integer. For a permutation  $\sigma$  of  $[n]$ , by  $\text{alt}_\sigma(X)$ , we denote the length of a longest alternating subsequence of non-zero signs in  $(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ , i.e.,  $\text{alt}((x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}))$ . In this terminology,  $\text{alt}(X)$  is the same as  $\text{alt}_I(X)$ , where  $I$  is the identity permutation. Now, we are in a position to introduce Tucker's lemma.

**Lemma A.** (*Tucker's lemma* [31]) Suppose that  $n$  is a positive integer and  $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ . Also, assume that for any signed set  $X \in \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\}$ , we have  $\lambda(-X) = -\lambda(X)$ . Then, there exist two signed sets  $X$  and  $Y$  such that  $X \preceq Y$  and also  $\lambda(X) = -\lambda(Y)$ .

There exist several interesting applications of Tucker's lemma in combinatorics. Among them, one can consider a combinatorial proof for Lovász-Kneser's Theorem by Matousek [21]. Also, there are various generalizations of Tucker's lemma. Next lemma is a combinatorial variant of  $Z_p$ -Tucker Lemma proved and modified in [34] and [24], respectively.

**Lemma B.** ( *$Z_p$ -Tucker Lemma*) Suppose that  $n, m, p$  and  $\alpha$  are non-negative integers, where  $m, n \geq 1$ ,  $m \geq \alpha \geq 0$ , and  $p$  is a prime number. Also, let

$$\begin{aligned} \lambda : (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\} &\longrightarrow \mathcal{Z}_p \times [m] \\ X &\longmapsto (\lambda_1(X), \lambda_2(X)) \end{aligned}$$

be a map satisfying the following properties:

- $\lambda$  is a  $Z_p$ -equivariant map, that is, for each  $\omega^j \in \mathcal{Z}_p$ , we have  $\lambda(\omega^j X) = (\omega^j \lambda_1(X), \lambda_2(X))$ ;
- for all  $X_1 \preceq X_2 \in (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ , if  $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$ , then  $\lambda_1(X_1) = \lambda_1(X_2)$ ;

- for all  $X_1 \preceq X_2 \preceq \dots \preceq X_p \in (Z_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ , if  $\lambda_2(X_1) = \lambda_2(X_2) = \dots = \lambda_2(X_p) \geq \alpha + 1$ , then the  $\lambda_1(X_i)$ 's are not pairwise distinct for  $i = 1, 2, \dots, p$ .

Then  $\alpha + (m - \alpha)(p - 1) \geq n$ .

Another interesting generalization of the Borsuk-Ulam theorem is Ky Fan's lemma [10]. This lemma has been used in some papers to study some coloring properties of graphs, see [5, 12].

**Lemma C.** (Tucker-Ky Fan's lemma [10]) Assume that  $m$  and  $n$  are positive integers and  $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$  satisfying the following properties:

1. for any  $X \in \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\}$ , we have  $\lambda(-X) = -\lambda(X)$  (a  $Z_2$ -equivariant map)
2. there are no any two signed sets  $X$  and  $Y$  such that  $X \preceq Y$  and  $\lambda(X) = -\lambda(Y)$ .

Then there are  $n$  signed sets  $X_1 \preceq X_2 \preceq \dots \preceq X_n$  such that  $\{\lambda(X_1), \dots, \lambda(X_n)\} = \{+a_1, -a_2, \dots, (-1)^{n-1}a_n\}$  where  $1 \leq a_1 < \dots < a_n \leq m$ . In particular  $m \geq n$ .

### 3 An Improvement of Dol'nikov-Kříž Theorem

For a hypergraph  $\mathcal{F} \subseteq 2^{[n]}$ , a permutation  $\sigma$  of  $[n]$  (a *linear ordering* of  $[n]$ ), and positive integers  $i$  and  $r \geq 2$ , set  $\text{alt}_{r,\sigma}(\mathcal{F}, i)$  to be the largest integer  $k$  such that there exists an  $X \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  with  $\text{alt}_\sigma(X) = k$  and that the chromatic number of hypergraph  $\text{KG}^r(\mathcal{F}|_X)$  is at most  $i - 1$ . Indeed,  $\text{alt}_{r,\sigma}(\mathcal{F}, 1)$  is the largest integer  $k$  such that there exists an  $X = (X^1, X^2, \dots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  with  $\text{alt}_\sigma(X) = k$  and none of the  $X^j$ 's contain any member of  $\mathcal{F}$ . For instance, one can see that  $\text{alt}_{r,I}(\binom{[n]}{k}_2) = r(k - 1) + 1$ . Also, hereafter,  $\text{alt}_{r,\sigma}(\mathcal{F}, 1)$  is denoted by  $\text{alt}_{r,\sigma}(\mathcal{F})$ . Now, set  $\text{alt}_r(\mathcal{F}, i) = \min\{\text{alt}_{r,\sigma}(\mathcal{F}, i); \sigma \in S_n\}$ . Also,  $\text{alt}_r(\mathcal{F}, i)$  is termed the  $i^{\text{th}}$  *alternation number* of  $\mathcal{F}$  (with respect to  $r$ ) and the *first alternation number* of  $\mathcal{F}$  is denoted by  $\text{alt}_r(\mathcal{F})$ . In this terminology, one can see that if  $i > \chi(\text{KG}^r(\mathcal{F}))$ , then  $\text{alt}_r(\mathcal{F}, i) = n$ .

For a hypergraph  $\mathcal{F} \subseteq 2^{[n]}$ , define  $M_r(\mathcal{F})$  to be the maximum size of a set  $T \subseteq [n]$  such that the induced hypergraph on  $\mathcal{F}|_T$ , is an  $r$ -colorable hypergraph. One can see that  $cd_r(\mathcal{F}) = n - M_r(\mathcal{F})$ . In view of Theorem A, we know  $\chi(\text{KG}^r(\mathcal{F})) \geq \frac{cd_r(\mathcal{F})}{r-1} = \frac{n - M_r(\mathcal{F})}{r-1}$ . In this section, we show that  $\chi(\text{KG}^r(\mathcal{F})) \geq \frac{n - \text{alt}_r(\mathcal{F})}{r-1}$ . In fact, Theorem A is an immediate consequence of this result. To see this, one can check that for any permutation  $\sigma$  of  $[n]$ , we have  $M_r(\mathcal{F}) \geq \text{alt}_{r,\sigma}(\mathcal{F})$ . According to the definition of  $\text{alt}_{r,\sigma}(\mathcal{F})$ , there is an  $X \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ , such that none of  $X^k$ 's ( $1 \leq k \leq r$ ) contain any member of  $\mathcal{F}$  and that  $\text{alt}_\sigma(X) = \text{alt}_{r,\sigma}(\mathcal{F})$ . Set  $T = \cup_{a=1}^r X^a$ . Obviously,  $(X^1, \dots, X^r)$  is a proper  $r$ -coloring of  $\mathcal{F}|_T$ . This implies that  $|X| \leq M_r(\mathcal{F})$ . On the other hand,  $\text{alt}_\sigma(X) \leq |X|$  and therefore

$$\chi(\text{KG}^r(\mathcal{F})) \geq \frac{n - \text{alt}_{r,\sigma}(\mathcal{F})}{r-1} \geq \frac{n - |X|}{r-1} \geq \frac{n - M_r(\mathcal{F})}{r-1} = \frac{cd_r(\mathcal{F})}{r-1}.$$

Assume that  $h$  is a proper coloring of  $H = \text{KG}^r(\mathcal{F})$  with colors  $\{1, 2, \dots, t\}$  where  $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ . For any subset  $B \subseteq [n]$ , we define  $\bar{h}(B) = \max\{h(S) : S \subseteq B, S \in \mathcal{F}\}$ , if there is no  $S \subseteq B$  where  $S \in \mathcal{F}$ , then set  $\bar{h}(B) = 0$ . For any  $X = (X^1, X^2, \dots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ , define  $\bar{h}(X) = \max\{\bar{h}(X^1), \bar{h}(X^2), \dots, \bar{h}(X^r)\}$ , i.e.,

$$\bar{h}(X) = \max\{h(A) : A \in \mathcal{F} \text{ \& } \exists j \in [r] \text{ s.t. } A \subseteq X^j\}.$$

Now, we are ready to improve Theorem A.

**Lemma 1.** *Assume that  $\mathcal{F} \subseteq 2^{[n]}$  is a hypergraph, and  $p$  is a prime number. For any positive integer  $i$  where  $i \leq \chi(\text{KG}^p(\mathcal{F})) + 1$ , we have*

$$\chi(\text{KG}^p(\mathcal{F})) \geq \frac{n - \text{alt}_p(\mathcal{F}, i)}{p-1} + i - 1.$$

**Proof.** Consider an arbitrary total ordering  $\leq$  on  $2^{[n]}$ . To prove the assertion, it is enough to show that for any  $\sigma \in S_n$ , we have  $\chi(\text{KG}^p(\mathcal{F})) \geq \frac{n - \text{alt}_{p, \sigma}(\mathcal{F}, i)}{p-1} + i - 1$ . Without loss of generality, we can suppose  $\sigma = I$ . Let  $\text{KG}^p(\mathcal{F})$  be properly colored with  $C$  colors  $\{1, 2, \dots, C\}$ . For any  $F \in \mathcal{F}$ , we denote its color by  $h(F)$ . Set  $\alpha = \text{alt}_{p, I}(\mathcal{F}, i)$  and  $m = \text{alt}_{p, I}(\mathcal{F}, i) + C - i + 1$ .

Now, define a map  $\lambda : (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\} \longrightarrow \mathcal{Z}_p \times [m]$  as follows

- If  $X \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \leq \text{alt}_{p, I}(\mathcal{F}, i)$ , set  $\lambda(X) = (\omega^j, \text{alt}_I(X))$ , where  $j$  is the index of set  $X^j$  containing the smallest integer ( $\omega^j$  is then the first non-zero term in  $X = (x_1, \dots, x_n)$ ).
- If  $X \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \geq \text{alt}_{p, I}(\mathcal{F}, i) + 1$ , in view of definition of  $\text{alt}_{p, I}(\mathcal{F}, i)$ , the chromatic number of  $\text{KG}^p(\mathcal{F}|_X)$  is at least  $i$ . Set  $\lambda(X) = (\omega^j, \bar{h}(X) - i + 1 + \alpha)$ , where  $j$  is a positive integer such that there is an  $A \in \mathcal{F}$  where  $A \subseteq X^j$ ,  $h(A) = \bar{h}(X)$ , and  $A$  is the biggest such a subset respect to the total ordering  $\leq$  (note that  $\bar{h}(X) \geq i$ ).

One can check that  $\lambda$  is a  $\mathcal{Z}_p$ -equivariant map from  $(\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  to  $\mathcal{Z}_p \times [m]$ .

Let  $X_1 \preceq X_2 \in (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ . If  $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$ , then the size of longest alternating subsequences of non-zero terms of  $X_1$  and  $X_2$  are the same. Therefore, the first non-zero terms of  $X_1$  and  $X_2$  are equal; and equivalently,  $\lambda_1(X_1) = \lambda_1(X_2)$ .

Assume that  $X_1 \preceq X_2 \preceq \dots \preceq X_p \in (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  such that  $\lambda_2(X_1) = \lambda_2(X_2) = \dots = \lambda_2(X_p) \geq \alpha + 1$ . According to the definition of  $\lambda$ , for each  $1 \leq a \leq p$ , there are  $F_a \in \mathcal{F}$  and  $j_a \in [p]$  such that  $F_a \subseteq X_a^{j_a}$  and  $\lambda_2(X_a) = h(F_a) + i - 1 + \alpha$ . This implies that  $h(F_1) = h(F_2) = \dots = h(F_p)$ . If  $|\{j_1, j_2, \dots, j_p\}| = p$ , then  $\{F_1, F_2, \dots, F_p\}$  is an edge in  $\text{KG}^p(\mathcal{F})$ . But, this is a contradiction because  $h$  is a proper coloring and  $h(F_1) = h(F_2) = \dots = h(F_p)$ .

Now, we can apply the  $\mathcal{Z}_p$ -Tucker Lemma and conclude that  $n \leq \text{alt}_{p, I}(\mathcal{F}, i) + (C - i + 1)(p - 1)$  and so  $C \geq \frac{n - \text{alt}_{p, I}(\mathcal{F}, i)}{p-1} + i - 1$ . ■



**Lemma 2.** Suppose that  $r, s$  and  $n$  are positive integers,  $X = (X^1, X^2, \dots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$ , and  $\sigma$  is a permutation of  $[n]$ . Also, assume that for each  $j \in [r]$ ,  $Y^{1j}, Y^{2j}, \dots, Y^{sj}$  are disjoint subsets of  $X^j$ . If we set

$$Y_j = (Y^{1j}, Y^{2j}, \dots, Y^{sj}) \in (\mathcal{Z}_s \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$$

and

$$Z = (Y^{11}, \dots, Y^{s1}, \dots, Y^{1r}, \dots, Y^{sr}) \in (\mathcal{Z}_{rs} \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\},$$

then  $\text{alt}_\sigma(Z) \geq \sum_{i=1}^r \text{alt}_{\sigma|_{X^i}}(Y_i).$

**Proof.** Without loss of generality, we can suppose  $\sigma = I$ . Also, let  $\text{alt}_I(X) = t$ . If  $x_{a_1}, x_{a_2}, \dots, x_{a_t}$  form an alternating subsequence of  $X$  ( $1 \leq a_1 < a_2 < \dots < a_t \leq n$ ), then the set  $\{a_1, a_2, \dots, a_t\}$  is called the *index set* of this alternating subsequence. Choose an alternating subsequence  $x_{a_1}, x_{a_2}, \dots, x_{a_t}$  of  $X$  such that  $a_1$  is the smallest integer in  $T = \cup_{j=1}^r X^j$  and that for each  $i \in [t]$ , there is a  $j_i \in [r]$  where  $[a_i, a_{i+1}) \cap T \subseteq X^{j_i}$ . For each  $j \in [r]$ , assume that  $P_j$  is a longest alternating subsequence of  $Y_j$ . Now, we present an alternating subsequence  $P$  of  $Z$ . Construct  $P$  such that for each  $i \in [t]$ ,  $P$  and  $P_{j_i}$  have the same index set in  $[a_i, a_{i+1})$ . It is straightforward to check that  $P$  is an alternating subsequence of  $Z$  and also,  $|P| = \sum_{i=1}^r \text{alt}_{I|_{X^i}}(Y_i).$  ■

Here, we extend Lemma 1 to any  $r$  uniform hypergraph ( $r$  is not necessarily prime) for the first alternation number.

**Lemma 3.** Let  $r$  and  $s$  be positive integers where  $r, s \geq 2$ . Also, assume that for any hypergraph  $\mathcal{H} \subseteq 2^{[n]}$ ,  $\chi(\text{KG}^r(\mathcal{H})) \geq \frac{n - \text{alt}_r(\mathcal{H})}{r-1}$  and  $\chi(\text{KG}^s(\mathcal{H})) \geq \frac{n - \text{alt}_s(\mathcal{H})}{s-1}$ . For any hypergraph  $\mathcal{F} \subseteq 2^{[n]}$ , we have  $\chi(\text{KG}^{rs}(\mathcal{F})) \geq \frac{n - \text{alt}_{rs}(\mathcal{F})}{rs-1}.$

**Proof.** It is enough to show that for any  $\sigma \in S_n$ ,  $\chi(\text{KG}^{rs}(\mathcal{F})) \geq \frac{n - \text{alt}_{rs, \sigma}(\mathcal{F})}{rs-1}$ . Without loss of generality, we can suppose  $\sigma = I$ . Let  $K = \chi(\text{KG}^{rs}(\mathcal{F}))$ . On the contrary, suppose

$$n - \text{alt}_{rs, I}(\mathcal{F}) > (rs - 1)K. \quad (1)$$

Define the hypergraph  $\mathcal{T} \subseteq 2^{[n]}$  as follows

$$\mathcal{T} = \left\{ N \subseteq [n] : |N| - \text{alt}_{s, I|_N}(\mathcal{F}|_N) > (s-1)K \right\}.$$

Now, according to the assumption of theorem and the definition of  $\mathcal{T}$ , for each  $N \in \mathcal{T}$ , we have

$$(s-1)\chi(\text{KG}^s(\mathcal{F}|_N)) \geq |N| - \text{alt}_{s, I|_N}(\mathcal{F}|_N) > (s-1)K.$$

Consequently,

$$\chi(\text{KG}^s(\mathcal{F}|_N)) > K \quad (2)$$

**Claim:**  $n - \text{alt}_{r,I}(\mathcal{T}) > (r-1)K$ .

Suppose, contrary to our claim, that  $n - \text{alt}_{r,I}(\mathcal{T}) \leq (r-1)K$  and so  $\text{alt}_{r,I}(\mathcal{T}) \geq n - (r-1)K$ . By definition of  $\text{alt}_{r,I}(\mathcal{T})$ , there is an  $X = (X^1, X^2, \dots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  such that  $\text{alt}_I(X) \geq n - (r-1)K$  and none of  $X^j$ 's contain any member of  $\mathcal{T}$ . In particular, none of them lie in  $\mathcal{T}$ . Therefore, by the definition of  $\mathcal{T}$ , we have

$$|X^i| - \text{alt}_{s,I|_{X^i}}(\mathcal{F}|_{X^i}) \leq (s-1)K.$$

It means  $|X^i| - (s-1)K \leq \text{alt}_{s,I|_{X^i}}(\mathcal{F}|_{X^i})$ . Therefore, for each  $j \in [r]$ , there are  $s$  disjoint sets  $Y^{j1}, \dots, Y^{js} \subseteq X^j$ , such that  $\text{alt}_{I|_{X^j}}(Y^{j1}, \dots, Y^{js}) \geq |X^j| - (s-1)K$  and none of them contain any member of  $\mathcal{F}|_{X^j}$ . In particular, none of them contain any member of  $\mathcal{F}$ . Set

$$Z = (Y^{11}, \dots, Y^{1s}, \dots, Y^{r1}, \dots, Y^{rs}) \in (\mathcal{Z}_{rs} \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}.$$

By Lemma 2,

$$\text{alt}_I(Z) \geq \sum_{j=1}^r \text{alt}_{I|_{X^j}}(Y^{j1}, \dots, Y^{js}) \geq \sum_{j=1}^r (|X^j| - (s-1)K) \geq \left( \sum_{j=1}^r |X^j| \right) - r(s-1)K.$$

Note that  $\sum_{j=1}^r |X^j| \geq \text{alt}_I(X)$  and thus,

$$\text{alt}_I(Z) \geq \text{alt}_I(X) - r(s-1)K \geq n - (r-1)K - r(s-1)K = n - (sr-1)K.$$

Since  $Z$  does not contain any member of  $\mathcal{F}$ , we get  $\text{alt}_{rs,I}(\mathcal{F}) \geq n - (sr-1)K$  which contradicts inequality (1). So we have proved the Claim.

By the assumption of theorem and the claim, we have

$$(r-1)\chi(\text{KG}^r(\mathcal{T})) \geq n - \text{alt}_{r,I}(\mathcal{T}) > (r-1)K$$

and so,

$$\chi(\text{KG}^r(\mathcal{T})) > K. \quad (3)$$

Now, consider a proper coloring  $h : \mathcal{F} \rightarrow [K]$  of  $\text{KG}^{rs}(\mathcal{F})$ . By inequality (2), in every  $N \in \mathcal{T}$ , there exists a color  $i \in [K]$  which has been assigned to  $s$  disjoint members of  $\mathcal{F}|_N$ . Now, define  $h' : \mathcal{T} \rightarrow [K]$  such that  $h'(N)$  is the maximum color which  $h$  assigns to  $s$  disjoint sets in  $\mathcal{F}|_N$ . Now, according to inequality (3), there are  $r$  sets  $N_1, \dots, N_r \in \mathcal{T}$  which are disjoint and from  $h'$  receive the same color  $i_0 = h'(N_j)$ . Thus we have  $rs$  sets  $F_{jk} \in \mathcal{F}$  such that  $F_{jk} \subseteq N_j$  and also, they are disjoint and  $h$  assigns them the same color which is a contradiction. ■

Next theorem is an immediate consequence of Lemma 1 (in the case  $i = 1$ ) and Lemma 3.

**Theorem 1.** For any hypergraph  $\mathcal{F} \subseteq 2^{[n]}$  and positive integer  $r \geq 2$ , we have

$$\chi(\text{KG}^r(\mathcal{F})) \geq \frac{n - \text{alt}_r(\mathcal{F})}{r-1}.$$



Theorem 1 in general is better than Dol'nikov-Kříž lower bound. Ziegler in [34, 35] showed that  $cd_r(\binom{[n]}{k}_t) = \max\{n - tr(k-1), 0\}$ . Therefore, Dol'nikov-Kříž Theorem implies that  $\chi(KG^r(n, k)_{2-stab}) \geq \frac{\max\{n-2r(k-1), 0\}}{r-1}$ . Although, one can easily see that  $alt_{r,I}(\binom{[n]}{k}_2) = r(k-1) + 1$  and thus by Theorem 1,

$$\chi(KG^r(n, k)_{2-stab}) \geq \frac{n - r(k-1) - 1}{r-1} > \frac{\max\{n - 2r(k-1), 0\}}{r-1}.$$

It is easy to see that  $alt_{2,I}(\binom{[n]}{k}_2, 2) = 2k - 1$ . Therefore, in view of Lemma 1 for  $i = 2$ , we have the next corollary.

**Corollary 1.** [27] *If  $n$  and  $k$  are positive integers where  $n \geq 2k$ , then we have  $\chi(KG^2(n, k)_{2-stab}) = n - 2k + 2$ .*

## 4 Multiple Kneser Graphs

Throughout this section, we assume that  $k, r, n$  and  $m$  are positive integers where  $r \geq 2$  and  $k \geq 1$ . Furthermore, suppose that  $\pi = (P_1, P_2, \dots, P_m)$  is a *partition* of  $[n]$  and  $\vec{s} = (s_1, s_2, \dots, s_m)$  is a positive integer vector where  $k \leq \sum_{i=1}^m s_i$  and for any  $1 \leq i \leq m$ , we have  $s_i \leq |P_i|$ . The *multiple Kneser hypergraph*  $KG^r(\pi; \vec{s}; k)$  is a hypergraph with the vertex set

$$V = \{A : A \subseteq P_1 \cup P_2 \cup \dots \cup P_m, |A| = k, \forall 1 \leq i \leq m; |A \cap P_i| \leq s_i\},$$

where  $\{A_1, \dots, A_r\}$  is an edge if  $A_1, A_2, \dots, A_r$  are pairwise disjoint. In the sequel, we determine the chromatic number of multiple Kneser hypergraphs provided that  $r = 2$  or for any  $1 \leq i \leq m$ , we have  $|P_i| \leq 2s_i$ . In this regard, we define the function  $f_{r,\pi}$  as follows

$$f_{r,\pi}(P_i) = \begin{cases} rs_i & \text{if } |P_i| \geq rs_i \\ |P_i| & \text{otherwise.} \end{cases}$$

Also, set

$$M_{r,\pi} = \max \left\{ rk - 1 + \sum_{j=1}^t (|P_{i_j}| - f_{r,\pi}(P_{i_j})) : t \in \mathbb{N} \quad \& \quad \sum_{j=1}^t f_{r,\pi}(P_{i_j}) \leq rk - 1 \right\}.$$

In the next theorem, we give an upper bound for the chromatic number of multiple Kneser hypergraph  $KG^r(\pi; \vec{s}; k)$  in terms of  $n$  and  $M_{r,\pi}$ .

**Lemma 4.** *Let  $k, r, n$ , and  $m$  be positive integers where  $r \geq 2$  and  $k \geq 1$ . Also, assume that  $\pi = (P_1, P_2, \dots, P_m)$  is a partition of  $[n]$  and  $\vec{s} = (s_1, s_2, \dots, s_m)$  is a positive integer vector where  $k \leq \sum_{i=1}^m s_i$ . We have*

$$\chi(KG^r(\pi; \vec{s}; k)) \leq \max\left\{1, \left\lceil \frac{n - M_{r,\pi}}{r-1} + 1 \right\rceil\right\}.$$

**Proof.** Without loss of generality, we can suppose that  $t$  is the greatest positive integer such that the value of  $M_{r,\pi}$  is attained and moreover we suppose that  $M_{r,\pi}$  is obtained by  $P_m, P_{m-1}, \dots, P_{m-t+1}$ , i.e.,  $M_{r,\pi} = rk - 1 + \sum_{j=m-t+1}^m (|P_j| - f_{r,\pi}(P_j))$  and  $\sum_{j=m-t+1}^m f_{r,\pi}(P_j) \leq rk - 1$ . In view of definition of  $M_{r,\pi}$ , for any  $1 \leq i \leq$

$m - t$ , one can see that  $rk - \sum_{j=m-t+1}^m f_{r,\pi}(P_j) \leq f_{r,\pi}(P_i)$ . If  $m - t = 0$ , then the chromatic number of multiple Kneser hypergraph is equal to one and there is nothing to prove. Hence, suppose  $m - t \geq 1$ . Consider  $L$  to be a subset of  $P_{m-t}$  of size

$rk - 1 - \sum_{j=m-t+1}^m f_{r,\pi}(P_j)$ . Set  $T = L \cup \left( \bigcup_{j=m-t+1}^m P_j \right)$ . Note that the size of

$C = \left( \bigcup_{j=1}^{m-t} P_j \right) \setminus L$  is  $n - M_{r,\pi}$ . For convenience, we assume  $C = \{1, 2, \dots, n - M_{r,\pi}\}$ .

Now, we present a proper coloring for  $\text{KG}^r(\pi; \vec{s}; k)$  using  $\left\lceil \frac{n - M_{r,\pi}}{r-1} \right\rceil + 1$  colors.

We show that all the vertices of  $\text{KG}^r(\pi; \vec{s}; k)$  which are subsets of  $T$  form an independent set. To see this, suppose therefore (reductio ad absurdum) that this is not the case and assume that  $A_1, A_2, \dots, A_r \in \mathcal{V}(\text{KG}^r(\pi; \vec{s}; k))$  form an edge in  $\text{KG}^r(\pi; \vec{s}; k)$  where  $A_1, A_2, \dots, A_r \subseteq T$ . According to the definition of  $\text{KG}^r(\pi; \vec{s}; k)$ , we have  $\sum_{i=1}^r |A_i \cap P_j| \leq f_{r,\pi}(P_j)$  for any  $m-t+1 \leq j \leq m$  and also  $\sum_{i=1}^r |A_i \cap L| \leq |L|$ .

But

$$\begin{aligned} rk = \left| \bigcup_{i=1}^r A_i \right| &= \left( \sum_{i=1}^r |A_i \cap L| \right) + \sum_{j=m-t+1}^m \sum_{i=1}^r |A_i \cap P_j| \\ &\leq |L| + \sum_{j=m-t+1}^m f_{r,\pi}(P_j) \\ &= (rk - 1) - \sum_{j=m-t+1}^m f_{r,\pi}(P_j) + \sum_{j=m-t+1}^m f_{r,\pi}(P_j) \end{aligned}$$

which is a contradiction.

Note that the size of  $C = \left( \bigcup_{j=1}^{m-t} P_j \right) \setminus L$  is  $n - M_{r,\pi}$ . Set  $b = \left\lceil \frac{n - M_{r,\pi}}{r-1} \right\rceil$ . Consider a partition  $(Q_1, Q_2, \dots, Q_b)$  of  $C$  such that  $r - 1 = |Q_1| = |Q_2| = \dots = |Q_{b-1}| \geq |Q_b| > 0$ .

Now, we present a proper coloring for  $\text{KG}^r(\pi; \vec{s}; k)$  using  $b + 1$  colors. As we mentioned, all the vertices of  $\text{KG}^r(\pi; \vec{s}; k)$  that are subsets of  $T$  form an independent set and therefore we can assign a color to all of them, e.g.,  $b + 1$ . Since every other vertex  $A$  has a non-empty intersection with  $C$ , we define the color of this vertex to be the minimum integer  $j$  such that  $A \cap Q_j \neq \emptyset$ . ■

In the sequel, we show that  $\text{alt}_2(\mathcal{V}(\text{KG}^2(\pi; \vec{s}; k))) = M_{2,\pi} - 1$ .

**Lemma 5.** Let  $k, m$  and  $n$  be positive integers and  $\vec{s} = (s_1, s_2, \dots, s_m)$  be a positive integer vector where  $k \leq \sum_{i=1}^m s_i$ . Also, assume that  $\pi = (P_1, P_2, \dots, P_m)$  is a partition

of  $[n]$ , where each  $P_i$  is a subset of  $|P_i|$  consecutive numbers. If  $X \in \{+1, 0, -1\}^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \geq M_{2,\pi}$ , then either  $X^{+1}$  or  $X^{-1}$  contains a  $k$ -subset  $A$  of  $[n]$  such that  $A$  is a vertex of  $\text{KG}^2(\pi; \vec{s}; k)$ .

**Proof.** Suppose that  $X \in \{+1, 0, -1\}^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \geq M_{2,\pi}$ . Consider an alternating subsequence of non-zero signs in  $X$  of length  $\text{alt}_I(X)$ . Define  $\text{alt}_I(X, P_j)$  to be the length of that part of this alternating subsequence of  $X$  lied in  $P_j$ . Also, let  $\text{alt}^+(X, P_j)$  (resp.  $\text{alt}^-(X, P_j)$ ) be the number of positive (resp. negative) signs of this alternating subsequence of  $X$  lied in  $P_j$ . One can see that  $\text{alt}_I(X, P_j) = \text{alt}^+(X, P_j) + \text{alt}^-(X, P_j)$  and  $|\text{alt}^+(X, P_j) - \text{alt}^-(X, P_j)| \leq 1$ . Assume that  $k^+ = \sum_{j=1}^m \min\{s_j, \text{alt}^+(X, P_j)\}$  and  $k^- = \sum_{j=1}^m \min\{s_j, \text{alt}^-(X, P_j)\}$ . To prove the lemma, it is enough to show that either  $k^+ \geq k$  or  $k^- \geq k$ . Suppose therefore (reductio ad absurdum) that this is not the case. So

$$I(X) := \sum_{j=1}^m \min\{s_j, \text{alt}^+(X, P_j)\} + \sum_{j=1}^m \min\{s_j, \text{alt}^-(X, P_j)\} \leq 2k - 2.$$

Since  $\text{alt}_I(X, P_j) = \text{alt}^+(X, P_j) + \text{alt}^-(X, P_j)$  and  $|\text{alt}^+(X, P_j) - \text{alt}^-(X, P_j)| \leq 1$ , one can see that

$$\min\{s_j, \text{alt}^+(X, P_j)\} + \min\{s_j, \text{alt}^-(X, P_j)\} \geq \min\{f_{2,\pi}(P_j), \text{alt}_I(X, P_j)\}.$$

Therefore,

$$\begin{aligned} 2k - 2 &\geq \sum_{j=1}^m \min\{f_{2,\pi}(P_j), \text{alt}_I(X, P_j)\} \\ &= \sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j) + \sum_{\{j: \text{alt}_I(X, P_j) < f_{2,\pi}(P_j)\}} \text{alt}_I(X, P_j). \end{aligned}$$

This means that  $\sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j) \leq 2k - 2$  and according to the definition of  $M_{2,\pi}$  we have

$$2k - 1 + \sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} (|P_j| - f_{2,\pi}(P_j)) \leq M_{2,\pi}$$

and therefore

$$2k - 1 - M_{2,\pi} + \sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} |P_j| \leq \sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j).$$

Now, we have

$$2k - 1 - M_{2,\pi} + \sum_{\{j: \text{alt}_I(X, P_j) \geq f_{2,\pi}(P_j)\}} |P_j| + \sum_{\{j: \text{alt}_I(X, P_j) < f_{2,\pi}(P_j)\}} \text{alt}_I(X, P_j) \leq I(X).$$

On the other hand,  $I(X) \leq 2k - 2$  and so

$$1 + alt_I(X) \leq 1 + \sum_{\{j: alt_I(X, P_j) \geq f_{2, \pi}(P_j)\}} |P_j| + \sum_{\{j: alt_I(X, P_j) < f_{2, \pi}(P_j)\}} alt_I(X, P_j) \leq M_{2, \pi},$$

which is a contradiction. ■

**Theorem 2.** *Let  $k, n$ , and  $m$  be positive integers where  $k \geq 1$ . Also, assume that  $\pi = (P_1, P_2, \dots, P_m)$  is a partition of  $[n]$  and  $\vec{s} = (s_1, s_2, \dots, s_m)$  is a positive integer vector where  $k \leq \sum_{i=1}^m s_i$ . We have*

$$\chi(KG^2(\pi; \vec{s}; k)) = \max\{1, n - M_{2, \pi} + 1\}.$$

**Proof.** To prove this theorem, according to Lemma 4, it is enough to show that  $\chi(KG^2(\pi; \vec{s}; k)) \geq n - (M_{2, \pi} - 1)$ . Define  $p_i = |P_i|$ ,  $q_0 = 0$ , and  $q_i = p_1 + \dots + p_i$ . Without loss of generality, we can suppose that for any  $1 \leq i \leq m$ ,  $P_i = \{q_{i-1} + 1, \dots, q_i\}$ . If  $M_{2, \pi} \geq n$ , then according to Lemma 4, the assertion holds. Therefore, we assume that  $M_{2, \pi} < n$ . Set  $\mathcal{F} = \mathcal{V}(KG^2(\pi; \vec{s}; k))$ . In view of Lemma 5, we have  $alt_I(\mathcal{F}) \geq M_{2, \pi} - 1$ . Consequently, by Theorem 1,

$$\chi(KG^2(\pi; \vec{s}; k)) \geq n - alt_{2, I}(\mathcal{F}) \geq n - (M_{2, \pi} - 1).$$
■

Note that Theorem 2 provide a generalization of Lovász-Kneser Theorem [19]. In fact, if we set  $|P_1| = |P_2| = \dots = |P_m| = 1$  and  $s_1 = s_2 = \dots = s_m = 1$  ( $\vec{s} = (1, 1, \dots, 1)$ ), then  $KG^2(\pi; \vec{s}; k) = KG^2(m, k)$ .

In [30], Tardif introduced the graph  $K_t^{k, m}$  and called it the *fractional multiple* of the complete graph  $K_t$ . This graph can be represented as follows. The vertices of  $K_t^{k, m}$  represent independent sets of size  $k$  in a disjoint union of  $m$  copies of  $K_t$ , and two of these are joined by an edge in  $K_t^{k, m}$  if they are disjoint. In [25, 26], the chromatic number of  $K_t^{k, m}$  was determined provided that  $t$  is even. It was shown that  $\chi(K_t^{k, m}) = t(m - k + 1)$  where  $t$  is an even integer and  $k \leq m$ . Although, the chromatic number of  $K_t^{k, m}$  for any odd integer  $t \geq 3$  was remained as an open problem. Moreover, it was conjectured in [25] that  $\chi(K_t^{k, m}) = t(m - k + 1)$  where  $t \geq 3$  is odd and  $k \leq m$ . In [26], it has been shown to prove  $\chi(K_t^{k, m}) = t(m - k + 1)$ , it suffices to show that  $\chi(K_3^{k, m}) = 3(m - k + 1)$ , since  $K_{2t+3}^{k, m}$  contains a complete join of  $K_{2t}^{k, m}$  and  $K_3^{k, m}$ .

Note that if we set  $|P_1| = |P_2| = \dots = |P_m| = t$  and  $s_1 = s_2 = \dots = s_m = 1$  ( $\vec{s} = (1, 1, \dots, 1)$ ), then  $KG^2(\pi; \vec{s}; k) = K_t^{k, m}$ . Therefore, in view of Theorem 2, we have the next corollary which gives an affirmative answer to the aforementioned conjecture [25].

**Corollary 2.** *Let  $t, k$  and  $m$  be positive integers where  $k \leq m$  and  $t \geq 2$ . Then  $\chi(K_t^{k, m}) = t(m - k + 1)$ .*

Alon et al. [2] determined the chromatic number of Kneser hypergraphs, i.e.,  $\chi(KG^r(n, k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . Here, we introduce a generalization of this result.

**Theorem 3.** *Let  $k, r, n$ , and  $m$  be positive integers where  $r \geq 2$  and  $k \geq 1$ . Also, assume that  $\pi = (P_1, P_2, \dots, P_m)$  is a partition of  $[n]$  and  $\vec{s} = (s_1, s_2, \dots, s_m)$  is a positive integer vector, where for each  $i \in [m]$ ,  $|P_i| \leq 2s_i$  and that  $k \leq \sum_{i=1}^m s_i$ . We have*

$$\chi(KG^r(\pi; \vec{s}; k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

**Proof.** Set  $p_i = |P_i|$ ,  $q_0 = 0$ , and  $q_i = p_1 + \dots + p_i$ . Without loss of generality, we can suppose that for any  $1 \leq i \leq m$ ,  $P_i = \{q_{i-1} + 1, \dots, q_i\}$ . One can check that  $M_{r,\pi} = rk - 1$ . Set  $\mathcal{F} = \mathcal{V}(KG^r(\pi; \vec{s}; k))$ . In view of Theorem 1 and Lemma 4, the proof is completed by showing that  $alt_{r,I}(\mathcal{F}) \leq M_{r,\pi} - r + 1 = r(k-1)$ . On the contrary, let  $X = (X^1, X^2, \dots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  such that  $alt_I(X) \geq r(k-1) + 1$  and none of  $X^i$ 's contain any vertex of  $KG^r(\pi; \vec{s}; k)$ . Consider an alternating subsequence of non-zero terms of  $X$  of length at least  $r(k-1) + 1$ . Define  $alt_I(X, P_j)$  to be the length of that part of this alternating subsequence lied in  $P_j$ . Also, for each  $\omega_i \in \mathcal{Z}_r$ , let  $alt(X, P_j, \omega_i)$  be the number of  $\omega_i$ 's of this alternating subsequence lied in  $P_j$ . One can see that  $alt_I(X, P_j) = \sum_{i=1}^r alt(X, P_j, \omega_i)$ . We can establish the theorem, if we prove there exists an  $\omega_i \in \mathcal{Z}_r$  such that

$$\sum_{j=1}^m \min \{s_j, alt(X, P_j, \omega_i)\} \geq k.$$

Suppose therefore (reductio ad absurdum) that this is not the case. Hence,

$$\sum_{i=1}^r \sum_{j=1}^m \min \{s_j, alt(X, P_j, \omega_i)\} \leq r(k-1).$$

Note that for any  $j \in [m]$ , we know  $|P_j| \leq 2s_j$  and so this implies that, for every  $j \in [m]$  and  $\omega_i \in \mathcal{Z}_r$ , we have  $\min \{s_j, alt(X, P_j, \omega_i)\} = alt(X, P_j, \omega_i)$ . Consequently,

$$alt_I(X) = \sum_{j=1}^m \sum_{i=1}^r \min \{s_j, alt(X, P_j, \omega_i)\} \leq r(k-1),$$

which is a contradiction. ■

## 5 Stable Kneser Graphs

In this section, we investigate the chromatic number of  $s$ -stable Kneser hypergraphs and almost  $s$ -stable Kneser hypergraphs. Next proposition was proved in [24] and here we present another proof for this result.

**Proposition 1.** Let  $k, n, r$ , and  $s$  be non-negative integers where  $n \geq sk$  and  $s \geq r \geq 2$ . We have

$$\chi(\text{KG}^r(n, k)_{s\text{-stab}}) \leq \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

**Proof.** Assume  $n = sq + t$  where  $0 < t \leq s$ . Set  $P_{i+1} = \{is+1, is+2, \dots, (i+1)s\}$  for  $0 \leq i \leq q-1$  and  $P_{q+1} = \{sq+1, sq+2, \dots, n\}$ . Also, define  $\pi = (P_1, P_2, \dots, P_{q+1})$  and  $\vec{s} = (1, 1, \dots, 1)$ . By Lemma 4, we have

$$\chi(\text{KG}^r(\pi; \vec{s}; k)) \leq \left\lceil \frac{n - M_{r,\pi}}{r-1} + 1 \right\rceil.$$

One can check that  $M_{r,\pi} = (k-1)s + r - 1$ . Therefore, since  $\text{KG}^r(n, k)_{s\text{-stab}}$  is a subgraph of  $\text{KG}^r(\pi, \vec{s}, k)$ , we have

$$\chi(\text{KG}^r(n, k)_{s\text{-stab}}) \leq \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

■

**Lemma 6.** Let  $r, s$  and  $p$  be positive integers where  $r \geq s \geq 2$  and  $p$  is a prime number. Assume that for any  $n \geq rk$ ,  $\chi(\text{KG}^r(n, k)_{s\text{-stab}}) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil$ . For any  $n \geq prk$ , we have  $\chi(\text{KG}^{pr}(n, k)_{s\text{-stab}}) = \left\lceil \frac{n - pr(k-1)}{pr-1} \right\rceil$ .

**Proof.** We endow  $2^{[n]}$  with an arbitrary total ordering  $\leq$ . Set  $L = \left\lceil \frac{n - pr(k-1)}{pr-1} \right\rceil$ . In view of Proposition 1, we know that  $\chi(\text{KG}^{pr}(n, k)_{s\text{-stab}}) \leq L = \left\lceil \frac{n - pr(k-1)}{pr-1} \right\rceil$ . On the contrary, suppose that there is an integer  $n \geq prk$  such that  $\chi(\text{KG}^{pr}(n, k)_{s\text{-stab}}) < L$ . Let  $h$  be a proper  $(L-1)$ -coloring of  $\text{KG}^{pr}(n, k)_{s\text{-stab}}$ . For a subset  $A \subseteq [n]$  where  $|A| \geq rk$ , the hypergraph  $\text{KG}^r(|A|, k)_{s\text{-stab}}$  can be considered as a subhypergraph of  $\text{KG}^r(A, k)_{s\text{-stab}}$ . Hence, by assumption, we have

$$\chi(\text{KG}^r(A, k)_{s\text{-stab}}) \geq \chi(\text{KG}^r(|A|, k)_{s\text{-stab}}) = \left\lceil \frac{|A| - r(k-1)}{r-1} \right\rceil.$$

Consequently, if  $|A| > (r-1)(L-1) + r(k-1)$ , then  $\chi(\text{KG}^r(A, k)_{s\text{-stab}}) > L-1$ . In this case, there are  $r$  pairwise disjoint vertices of  $\text{KG}^r(A, k)_{s\text{-stab}}$  such that  $h$  assigns the same color to all of them. Set  $m = p((r-1)(L-1) + r(k-1)) + L-1$ . Now, we are going to introduce a map  $\lambda : (Z_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\} \rightarrow Z_p \times [m]$ . First, note that if  $X \in (Z_p \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}(X) > p((r-1)(L-1) + r(k-1))$ , then there is an  $1 \leq i \leq p$  such that  $|X^i| > (r-1)(L-1) + r(k-1)$ . So,  $\chi(\text{KG}^r(X^i, k)_{s\text{-stab}}) > L-1$ ; and therefore, there are  $r$  pairwise disjoint vertices  $B_1, B_2, \dots, B_r$  of  $\text{KG}^r(X^i, k)_{s\text{-stab}}$  such that  $h(B_1) = \dots = h(B_r) = c$ . Set  $\bar{h}(X)$  to be the greatest such a color  $c$ . Precisely,

$$\bar{h}(X) = \max \left\{ c : \exists i, B_1, \dots, B_r \in \binom{X^i}{s}, B_i \cap B_j = \emptyset, h(B_1) = \dots = h(B_r) = c \right\}$$

Now, define  $\lambda(X)$  as follows

- if  $\text{alt}_I(X) \leq p((r-1)(L-1) + r(k-1))$ , set  $\lambda(X) = (\omega^j, \text{alt}(X))$  such that  $j$  is the least integer in  $\bigcup_{k=1}^p X^k$ .
- if  $\text{alt}_I(X) \geq p((r-1)(L-1) + r(k-1)) + 1$ , define  $\lambda(X) = (\omega^i, p(r-1)(L-1) + pr(k-1) + \bar{h}(X))$  such that there are  $r$  pairwise disjoint vertices  $B_1, B_2, \dots, B_r$  for which  $h(B_1) = h(B_2) = \dots = h(B_r) = \bar{h}(X)$ ,  $\bigcup_{k=1}^r B_k \subset X^i$  and  $X^i$  is the biggest such a component of  $X = (X^1, X^2, \dots, X^p)$  respect to the ordering  $\leq$ .

One can see that  $\lambda$  satisfies the conditions of  $Z_p$ -Tucker Lemma and therefore we should have

$$D = p(r-1)(L-1) + pr(k-1) + (L-1)(p-1) \geq n.$$

But,

$$\begin{aligned} D &= (pr-1)(L-1) + pr(k-1) \\ &\leq (pr-1)\left(\frac{n-pr(k-1)+pr-2}{pr-1} - 1\right) + pr(k-1) \\ &= n-1, \end{aligned}$$

which is a contradiction. ■

It was proved in [3] that for  $r = 2^j$ ,  $\chi(\text{KG}^r(n, k)_{r\text{-stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . Next corollary is an immediate consequence of this result and Lemma 6.

**Corollary 3.** Assume that  $a, k, n$  and  $r$  are positive integers. If we have  $2^a | r$ , then  $\chi(\text{KG}^r(n, k)_{2^a\text{-stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ .

**Theorem 4.** For any positive integers  $k, n$  and  $r$  where  $n \geq rk$ , if  $n \not\equiv^{r-1} k$  or  $r$  is an even integer, then  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ .

**Proof.** In view of Corollary 3, if  $r$  is an even integer, then there is nothing to prove. One can see that  $\text{KG}^r(n, k)_{2\text{-stab}}$  is a subgraph of  $\text{KG}^r(n, k)$ . This implies that  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}) \leq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ . Hence, it is sufficient to show that if  $n \not\equiv^{r-1} k$ , then  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}) \geq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ .

Assume that  $X \in (Z_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \geq r(k-1) + 2$ . One can see that there exists at least an  $X^i$  (for some  $1 \leq i \leq r$ ) containing some vertex of  $\text{KG}^r(n, k)_{2\text{-stab}}$ . Therefore,  $\text{alt}_{r,I}(\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)_2) \leq r(k-1) + 1$ . By Theorem 1, we have

$$\chi(\text{KG}^r(n, k)_{2\text{-stab}}) \geq \left\lceil \frac{n - \text{alt}_{r,I}(\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)_2)}{r-1} \right\rceil \geq \left\lceil \frac{n - r(k-1) - 1}{r-1} \right\rceil$$

One can check that  $\left\lceil \frac{n-r(k-1)-1}{r-1} \right\rceil = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$  provided that  $n \not\equiv^{r-1} k$ . ■



In view of  $Z_p$ -Tucker Lemma, Meunier [24] proved that, for any positive integer  $r$  and any  $n \geq kp$ , the chromatic number of  $\text{KG}^r(n, k)_{2\text{-stab}}^\sim$  is the same as the chromatic number of  $\text{KG}^r(n, k)$ , namely that is equal to  $\left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ .

**Theorem B.** [24] *For any  $r \geq 2$ , we have  $\chi(\text{KG}^r(n, k)_{2\text{-stab}}^\sim) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ .*

**Proof.** We proceed analogously to the proof of Theorem 4. Note that if  $X \in (Z_r \cup \{0\})^n \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}_I(X) \geq r(k-1) + 1$ , then there exists at least an  $X^i$  (for some  $1 \leq i \leq r$ ) containing some vertex of  $\text{KG}^r(n, k)_{2\text{-stab}}^\sim$ . Therefore,  $\text{alt}_{r,I}(\binom{[n]}{k}_2^\sim) \leq r(k-1)$ . By Theorem 1, we have

$$\chi(\text{KG}^r(n, k)_{2\text{-stab}}^\sim) \geq \left\lceil \frac{n - \text{alt}_{r,I}(\binom{[n]}{k}_2^\sim)}{r-1} \right\rceil \geq \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

This completes the proof. ■

## 6 Colorful Graphs

We say that a graph is *completely multicolored* in a coloring (*colorful*) if all its vertices receive different colors.

**Theorem C.** [29] *Let  $\mathcal{F}$  be a hypergraph and  $r = \text{cd}_2(\mathcal{F})$ . Then any proper coloring of  $\text{KG}(\mathcal{F})$  with colors  $\{1, 2, \dots, k\}$  ( $k$  arbitrary) must contain a completely multicolored complete bipartite graph  $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$  such that the  $r$  different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

We should mention that there are several versions of Theorem C in terms of some other parameters in graphs, see [6, 11, 22, 28]. The aforementioned theorem presents a lower bound for *local chromatic number* of a graph which is the minimum number of colors that must appear within distance 1 of a vertex, for more about local chromatic number, see [8, 28]. Next theorem provides a generalization of Theorem C in terms of alternation number of graphs.

**Theorem 5.** *Let  $\mathcal{F} \subseteq 2^{[n]}$  be a hypergraph and  $r = n - \text{alt}_2(\mathcal{F})$ . Then any proper coloring of  $\text{KG}(\mathcal{F})$  with colors  $\{1, 2, \dots, k\}$  ( $k$  arbitrary) must contain a completely multicolored complete bipartite graph  $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$  such that the  $r$  different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

**Proof.** Without loss of generality, we can suppose that  $\text{alt}_2(\mathcal{F}) = \text{alt}_{2,I}(\mathcal{F})$ , where  $I$  the identity permutation on  $[n]$ . First, assume that  $M = \text{alt}_{2,I}(\mathcal{F})$  is an even integer. Consider an arbitrary total ordering  $\leq$  on the power set of  $[n]$  that refines the partial ordering according to size. In other words, if  $|A| < |B|$ , then  $A \leq B$ , and sets of the same size can be ordered arbitrary, e.g., lexicographically. Assume that  $h$  is a proper coloring of  $G = \text{KG}(\mathcal{F})$  with  $k$  colors  $\{1, 2, \dots, k\}$ . Now, we construct a map  $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$  where  $m = M + k$ . For  $X \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$ , set  $\lambda(X)$  as follows

- If  $\text{alt}_I(X) \leq \text{alt}_{2,I}(\mathcal{F})$ , we define  $\lambda(X) = \pm \text{alt}_I(X)$ , where the sign is determined by the sign of the first element (with respect to the permutation  $I$ ) of the longest alternating subsequence of  $X$  (which is actually the first non-zero term of  $X$ ).
- If  $\text{alt}_I(X) \geq \text{alt}_{2,I}(\mathcal{F}) + 1$ , in view of definition of  $\text{alt}_I(\mathcal{F})$ , either  $X^+$  or  $X^-$  contains a member of  $\mathcal{F}$ . Define  $c = \max\{h(F) : F \in \mathcal{F}_X\}$ . Assume that  $F$  is a member of  $\mathcal{F}_X$  such that  $h(F) = c$ . Set  $\lambda(X) = \pm(h(F) + M)$ , where the sign is positive if  $F \subseteq X^+$  and negative if  $F \subseteq X^-$ .

It is straightforward to see that  $\lambda : \{-, 0, +\}^n \setminus \{(0, 0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$  satisfies the conditions of Tucker-Ky Fan's Lemma. Therefore, by Tucker-Ky Fan's Lemma, there are  $n$  signed sets  $X_1 \preceq X_2 \preceq \dots \preceq X_n$  such that  $\{\lambda(X_1), \dots, \lambda(X_n)\} = \{c_1, -c_2, c_3, \dots, (-1)^{n-1}c_n\}$  where  $1 \leq c_1 < c_2 < \dots < c_n \leq m$ .

For any  $1 \leq i \leq n$ , set  $|X_i| = |X_i^+ \cup X_i^-|$ . Since  $1 \leq |X_1| < |X_2| < \dots < |X_n| \leq n$ , we have  $|X_i| = i$ . Note that  $|\lambda|$  is a monotone function; and therefore,  $\lambda(X_i) = (-1)^{i-1}c_i$ . This observation concludes that  $|X_i^+| = \lceil \frac{i}{2} \rceil$  and  $|X_i^-| = \lfloor \frac{i}{2} \rfloor$ . In particular,  $|X_M^+| = \frac{M}{2}$  and  $|X_M^-| = \frac{M}{2}$ .

Note that for  $i \geq M + 1$ , we have  $|\lambda(X_i)| = \bar{h}(X_i) + M$  and this implies that

- if  $i$  is even, then  $\bar{h}(X_i^-) = c_i$
- if  $i$  is odd, then  $\bar{h}(X_i^+) = c_i$

Now, for any  $i = M + 2l \in \{M + 1, M + 2, \dots, n\}$ , there is an  $F_l \in \mathcal{F}$  such that  $F_l \subseteq X_i^- \subseteq X_n^-$  and  $h(F_l) = c_{M+2l}$ . Also, for any  $i = M + 2l - 1 \in \{M + 1, M + 2, \dots, n\}$ , there is a  $G_l \in \mathcal{F}$  such that  $F_l \subseteq X_i^+ \subseteq X_n^+$  and  $h(G_l) = c_{M+2l-1}$ . Since  $X_n^+ \cap X_n^- = \emptyset$ , the induced subgraph on vertices

$$\left\{ F_1, F_2, \dots, F_{\lfloor \frac{n - \text{alt}_2(\mathcal{F})}{2} \rfloor} \right\} \cup \left\{ G_1, G_2, \dots, G_{\lceil \frac{n - \text{alt}_2(\mathcal{F})}{2} \rceil} \right\}$$

is a complete bipartite graph which is the desired subgraph.

Now, assume that  $M$  is an odd integer. One can consider  $\mathcal{F}$  as a subset of  $2^{[n+1]}$ . Note that  $\text{alt}_2(\mathcal{F}) = M + 1$  is even integer and still  $n - M = (n + 1) - (M + 1)$ . Therefore, a similar proof works when  $M$  is an odd integer.  $\blacksquare$

Suppose that  $p$  and  $q$  are positive integers where  $p \geq 2q$  and  $G$  is a graph. A  $(p, q)$ -coloring of  $G$  is a mapping  $h : \mathcal{V}(G) \rightarrow \{0, 1, \dots, p - 1\}$  such that for any edge  $xy \in E(G)$ , we have  $q \leq |h(x) - h(y)| \leq p - q$ . The *circular chromatic number* of  $G$  is defined as follows

$$\chi_c(G) = \inf \left\{ \frac{p}{q} : G \text{ admits a } (p, q) \text{-coloring} \right\}$$

It is well-known [32] that  $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ . For more on circular chromatic number, see [32, 33].

The problem whether a graph has the same chromatic number and circular chromatic number has received attention, see [1, 23, 28, 32, 33]. For a  $t$ -coloring  $h$

of  $G$ , a cycle  $C = (v_0, v_1, \dots, v_{n-1}, v_0)$  is called *tight* if  $h(v_{i+1}) \stackrel{t}{=} h(v_i) + 1$  for  $i = 0, 1, \dots, n-1$ , where the indices of the vertices are modulo  $n$ . It is known [32] that for a positive integer  $t$ ,  $\chi_c(G) = t$  if and only if  $G$  is  $t$ -colorable and every  $t$ -coloring of  $G$  has a tight cycle.

Assume that  $\mathcal{F} \subseteq 2^{[n]}$  and  $\chi(\text{KG}^2(\mathcal{F})) = n - \text{alt}_2(\mathcal{F})$ . Previous theorem implies that, for any  $\chi(\text{KG}^2(\mathcal{F}))$ -coloring of  $\text{KG}^2(\mathcal{F})$ , there is a colorful complete bipartite graph  $K_{\lceil \frac{\chi(\text{KG}^2(\mathcal{F}))}{2} \rceil, \lfloor \frac{\chi(\text{KG}^2(\mathcal{F}))}{2} \rfloor}$ . This result implies that  $\chi_c(\text{KG}^2(\mathcal{F})) = \chi(\text{KG}^2(\mathcal{F}))$  provided that  $\chi(\text{KG}^2(\mathcal{F}))$  is an even integer.

**Corollary 4.** *Assume that  $\mathcal{F} \subseteq 2^{[n]}$  and  $\chi(\text{KG}^2(\mathcal{F})) = n - \text{alt}_2(\mathcal{F})$ . If  $\chi(\text{KG}^2(\mathcal{F}))$  is an even integer, then  $\chi_c(\text{KG}^2(\mathcal{F})) = \chi(\text{KG}^2(\mathcal{F}))$ .*

It has been conjectured in [14] that any Kneser graph has the same chromatic number and circular chromatic number. This conjecture has been studied in several papers, see [1, 4, 5, 13, 14, 23, 28]. Finally, Chen [5] completely proved this conjecture by using Fan's Lemma in an innovative way. Next, a shorter proof was presented in [4]. For  $\mathcal{F} = \mathcal{V}(\text{KG}^2(\pi; \vec{s}; k))$ , in view of the proof of Theorem 2, we have  $\text{alt}_2(\mathcal{F}) = M_{2,\pi} - 1$  and therefore,

$$\chi(\text{KG}^2(\pi; \vec{s}; k)) = \max\{1, n - \text{alt}_2(\mathcal{F})\}.$$

Next corollary is a consequence of Corollary 4.

**Corollary 5.** *Let  $k, n$ , and  $m$  be positive integers where  $k \geq 1$ . Also, assume that  $\pi = (P_1, P_2, \dots, P_m)$  is a partition of  $[n]$  and  $\vec{s} = (s_1, s_2, \dots, s_m)$  is a positive integer vector where  $k \leq \sum_{i=1}^m s_i$ . If  $\chi(\text{KG}^2(\pi; \vec{s}; k))$  is an even integer, then  $\chi_c(\text{KG}^2(\pi; \vec{s}; k)) = \chi(\text{KG}^2(\pi; \vec{s}; k))$ .*

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